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## Leibniz algebroid associated with a Nambu–Poisson structure

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**Abstract.** The notion of a Leibniz algebroid is introduced, and it is shown that each Nambu–Poisson manifold has associated a canonical Leibniz algebroid. This fact permits one to define the modular class of a Nambu–Poisson manifold as an appropriate cohomology class, extending the well known modular class of Poisson manifolds.

### 1. Introduction

A Lie algebroid is a natural generalization of the notion of Lie algebra, and also of the tangent bundle of a manifold. There are many other interesting examples, for instance, the cotangent bundle of any Poisson manifold possesses a natural Lie algebroid structure. Roughly speaking, a Lie algebroid over a manifold  $M$  is a vector bundle  $E$  over  $M$  such that its space of sections  $\Gamma(E)$  has a structure of Lie algebra plus a mapping (the anchor map) from  $E$  onto  $TM$  which provides a Lie algebra homomorphism from  $\Gamma(E)$  into the Lie algebra of vector fields  $\mathfrak{X}(M)$ .

The action of  $\Gamma(E)$  on  $C^\infty(M, \mathbb{R})$  defines the Lie algebroid cohomology of  $M$ . For a Poisson manifold  $M$ , the associated Lie algebroid is just the triple  $(T^*M, \llbracket \ , \ \rrbracket', \#)$ , where  $\llbracket \ , \ \rrbracket'$  is the bracket of 1-forms and  $\#$  is the mapping from 1-forms into tangent vectors defined by the Poisson tensor. For an oriented Poisson manifold  $M$  and its associated Lie algebroid, Weinstein [29, 30] has defined the so-called modular class of  $M$ , which is an element of the corresponding Lie algebroid cohomology (in fact, an element of the Lichnerowicz–Poisson cohomology space  $H_{LP}^1(M)$ ). The modular class  $X_\nu$  is defined as the operator which assigns to each function  $f$  the divergence with respect to  $\nu$  of its Hamiltonian vector field  $X_f$ , where  $\nu$  is a volume form on  $M$ . A direct computation shows that the modular class of a symplectic manifold is null. Indeed, the vanishing of the modular class of a Poisson manifold is closely related with its regularity. Moreover, it was proved by Xu [31] (see also [4, 9]) that the canonical homology is dual to the Lichnerowicz–Poisson cohomology for unimodular Poisson structures, that is, for Poisson structures with null modular class. Also, it should be remarked that the modular class was the tool recently used by Dufour and Haraki [8] and by Liu and Xu [16] to classify quadratic Poisson structures.

Our interest is to extend the above results for Nambu–Poisson manifolds. The concept of a Nambu–Poisson structure was introduced by Takhtajan [25] in order to find an axiomatic formalism for the  $n$ -bracket operation

$$\{f_1, \dots, f_n\} = \det \left( \frac{\partial f_i}{\partial x_j} \right)$$

proposed by Nambu [24] to generalize Hamiltonian mechanics (see also [2, 5, 6, 10]). A Nambu–Poisson manifold is a manifold  $M$  endowed with a skew-symmetric  $n$ -bracket of functions  $\{ \dots \}$  satisfying the Leibniz rule and the fundamental identity

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\}$$

for all  $f_1, \dots, f_{n-1}, g_1, \dots, g_n \in C^\infty(M, \mathbb{R})$ . The local and global structure of a Nambu–Poisson manifold were elucidated in recent papers [1, 11, 14, 22, 23]. Indeed, a Nambu–Poisson manifold of order greater than two consists of pieces which are volume manifolds, in the same way that a Poisson manifold is made of symplectic pieces. Recently, an interesting recursive characterization of Nambu–Poisson structures was obtained in [12].

In this paper we introduce the notion of a Leibniz algebroid—a natural generalization of a Lie algebroid. The notion of Leibniz algebra was recently introduced by Loday [17, 18] (see also [19]) as a non-commutative version of Lie algebras. Indeed, a Leibniz algebra is a real vector space  $\mathfrak{g}$  endowed with a  $\mathbb{R}$ -bilinear mapping  $\{ \cdot \}$  satisfying the Leibniz identity

$$\{a_1, \{a_2, a_3\}\} - \{\{a_1, a_2\}, a_3\} - \{a_2, \{a_1, a_3\}\} = 0$$

for all  $a_1, a_2, a_3 \in \mathfrak{g}$ . If the bracket is skew-symmetric we recover the notion of Lie algebra. Next, the notion of a Leibniz algebroid can be introduced in the same way as that for the case of Lie algebroids. One of the main results of the present paper is to associate a Leibniz algebroid to each Nambu–Poisson manifold  $M$ . The Leibniz algebroid attached to  $M$  is just the triple  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$ , where  $\llbracket \cdot, \cdot \rrbracket : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \rightarrow \Omega^{n-1}(M)$  is the bracket of  $(n-1)$ -forms defined by

$$\llbracket \alpha, \beta \rrbracket = \mathcal{L}_{\# \alpha} \beta + (-1)^n (i(\alpha) \Lambda) \beta$$

for  $\alpha, \beta \in \Omega^{n-1}(M)$ , and  $\# : \wedge^{n-1}(T^*M) \rightarrow TM$  is the homomorphism of vector bundles given by  $\#(\beta) = i(\beta) \Lambda$ . Here  $\Lambda$  is the Nambu–Poisson  $n$ -vector. In addition, it is proved that the only non-null Nambu–Poisson structures of order greater than two on an oriented manifold  $M$  of dimension  $m$ , with  $m \geq 3$ , such that its Leibniz algebroid is a Lie algebroid are those defined by non-null  $m$ -vectors.

As in the case of Poisson manifolds, we define the modular class of an oriented Nambu–Poisson  $m$ -dimensional manifold  $M$  of order  $n$ . Indeed, if  $\nu$  is a volume form on  $M$  then the mapping

$$\mathcal{M}_\nu : C^\infty(M, \mathbb{R}) \times \dots^{(n-1)} \dots \times C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$$

given by

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \nu = \mathcal{M}_\nu(f_1, \dots, f_{n-1}) \nu$$

is an  $(n-1)$ -vector on  $M$ , where  $X_{f_1 \dots f_{n-1}} = \#(df_1 \wedge \dots \wedge df_{n-1})$  is the Hamiltonian vector field associated with the functions  $f_1, \dots, f_{n-1}$ . Next, the mapping

$$\mathcal{M}_\Lambda : \Omega^{n-1}(M) \rightarrow C^\infty(M, \mathbb{R}) \quad \alpha \mapsto i(\alpha) \mathcal{M}_\nu$$

defines a 1-cocycle in the Leibniz cohomology complex associated with the Leibniz algebroid  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$ . The cohomology class  $[\mathcal{M}_\Lambda] \in H^1(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$  does not depend on the chosen volume form and it is called the modular class of  $M$ . As in the case of Poisson manifolds it is proved that the modular class of a volume manifold is null. In a forthcoming paper we will investigate the role played for the modular class in the problem of classification of Nambu–Poisson manifolds. We are also investigating the existence of a dual homology to the Leibniz algebroid cohomology, which would be related with the vanishing of the modular class (see [4, 9, 31] for the case of Lie algebroids and Poisson manifolds).

## 2. Preliminaries

All the manifolds considered in this paper are assumed to be connected.

### 2.1. Nambu–Poisson structures

Let  $M$  be a differentiable manifold of dimension  $m$ . Denote by  $\mathfrak{X}(M)$  the Lie algebra of vector fields on  $M$ , by  $C^\infty(M, \mathbb{R})$  the algebra of  $C^\infty$  real-valued functions on  $M$  and by  $\Omega^k(M)$  the space of  $k$ -forms on  $M$ .

An almost-Poisson bracket of order  $n$  ( $n \leq m$ ) on  $M$  (see [14]) is an  $n$ -linear mapping  $\{, \dots, \} : C^\infty(M, \mathbb{R}) \times \dots^{(n)} \dots \times C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$  satisfying the following properties:

(1) *Skew-symmetry*:

$$\{f_1, \dots, f_n\} = (-1)^{\varepsilon(\sigma)} \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\} \tag{2.1}$$

for all  $f_1, \dots, f_n \in C^\infty(M, \mathbb{R})$  and  $\sigma \in \text{Symm}(n)$ , where  $\text{Symm}(n)$  is a symmetric group of  $n$  elements and  $\varepsilon(\sigma)$  is the parity of the permutation  $\sigma$ .

(2) *Leibniz rule*:

$$\{f_1 g_1, f_2, \dots, f_n\} = f_1 \{g_1, f_2, \dots, f_n\} + g_1 \{f_1, f_2, \dots, f_n\} \tag{2.2}$$

for all  $f_1, \dots, f_n, g_1 \in C^\infty(M, \mathbb{R})$ .

If  $\{, \dots, \}$  is an almost-Poisson bracket of order  $n$  then we define a skew-symmetric tensor  $\Lambda$  of type  $(n, 0)$  ( $n$ -vector) as follows:

$$\Lambda(df_1, \dots, df_n) = \{f_1, \dots, f_n\}$$

for  $f_1, \dots, f_n \in C^\infty(M, \mathbb{R})$ . Conversely, given an  $n$ -vector on  $M$ , the above formula defines an almost-Poisson bracket of order  $n$ . The pair  $(M, \Lambda)$  is called a *generalized almost-Poisson manifold of order  $n$* .

If  $\wedge^{n-1}(T^*M)$  is the vector bundle of the  $(n - 1)$ -forms on  $M$  then  $\Lambda$  induces a homomorphism of vector bundles

$$\# : \wedge^{n-1}(T^*M) \longrightarrow TM$$

by defining

$$\#(\beta) = i(\beta)\Lambda(x) \tag{2.3}$$

for  $\beta \in \wedge^{n-1}(T_x^*M)$  and  $x \in M$ , where  $i(\beta)$  is the contraction by  $\beta$ . Denote also by

$$\# : \Omega^{n-1}(M) \rightarrow \mathfrak{X}(M)$$

the homomorphism of  $C^\infty(M, \mathbb{R})$ -modules given by

$$\#(\alpha)(x) = \#(\alpha(x)) \tag{2.4}$$

for all  $\alpha \in \Omega^{n-1}(M)$  and  $x \in M$ . Then, if  $f_1, \dots, f_{n-1}$  are  $n - 1$  functions on  $M$ , we define a vector field

$$X_{f_1 \dots f_{n-1}} = \#(df_1 \wedge \dots \wedge df_{n-1}) \tag{2.5}$$

which is called the *Hamiltonian vector field* associated with the Hamiltonian functions  $f_1, \dots, f_{n-1}$ . From (2.5) it follows that

$$X_{f_1 \dots f_{n-1}}(f_n) = \{f_1, \dots, f_{n-1}, f_n\}. \tag{2.6}$$

A richer structure, related with interesting dynamical problems, can be considered adding to the almost-Poisson bracket  $\{, \dots, \}$  the following integrability condition (*fundamental identity*):

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\} \tag{2.7}$$

for all  $f_1, \dots, f_{n-1}, g_1, \dots, g_n$  functions on  $M$ . In this case,  $\{, \dots, \}$  is called a *Nambu–Poisson bracket* and  $(M, \Lambda)$  is a *Nambu–Poisson manifold of order  $n$*  (see [25]).

In fact, an  $n$ -vector  $\Lambda$  on  $M$  defines a Nambu–Poisson structure if the Hamiltonian vector fields are derivations of the algebra  $(C^\infty(M, \mathbb{R}) \times \dots \times C^\infty(M, \mathbb{R}), \{, \dots, \})$  or equivalently, every Hamiltonian vector field  $X_{f_1 \dots f_{n-1}}$  is an infinitesimal automorphism of  $\Lambda$ , that is,

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \Lambda = 0.$$

**Examples 2.1.** (a) The *Poisson manifolds* are just the Nambu–Poisson manifolds of order two [15, 26, 28].

Another examples of Nambu–Poisson manifolds are the following.

(b) Let  $N$  be an oriented  $m$ -dimensional manifold and choose a volume form  $\nu_N$  on  $N$ . Given  $m$  functions  $f_1, \dots, f_m$  on  $N$ , we define its  $m$ -bracket by the formula

$$df_1 \wedge \dots \wedge df_m = \{f_1, \dots, f_m\} \nu_N. \tag{2.8}$$

It is not hard to prove that it is a Nambu–Poisson bracket (see [11]). Denote by  $\Lambda_{\nu_N}$  the  $m$ -vector associated with this bracket. Note that for the Nambu–Poisson structure  $\Lambda_{\nu_N}$  the homomorphism  $\# : \Omega^{m-1}(N) \rightarrow \mathfrak{X}(N)$  is an isomorphism. Furthermore, if  $\Lambda$  is a Nambu–Poisson structure of order  $m$  and  $\Lambda \neq 0$  at every point then there exists a volume form  $\nu$  on  $N$  such that  $\Lambda = \Lambda_{\nu_N}$  (see [14]).

(c) Let  $\Lambda_N$  be an arbitrary  $m$ -vector on an oriented  $m$ -dimensional manifold  $N$  with volume form  $\nu_N$ . Then, there exists a function  $f \in C^\infty(N, \mathbb{R})$  such that  $\Lambda_N = f \Lambda_{\nu_N}$ . Moreover, if  $f_1, \dots, f_{m-1}$  are  $m - 1$  functions on  $N$  and  $X_{f_1 \dots f_{m-1}}$  is the corresponding Hamiltonian vector field with respect to the  $m$ -vector  $\Lambda_N$ , it follows that  $\mathcal{L}_{X_{f_1 \dots f_{m-1}}} \Lambda_N = 0$ . Thus, we deduce that  $(N, \Lambda_N)$  is a Nambu–Poisson manifold of order  $m$ .

(d) If  $V$  is a  $k$ -dimensional differentiable manifold,  $\Lambda_N$  induces an  $m$ -vector  $\Lambda$  on the product  $N \times V$  and  $(N \times V, \Lambda)$  is a Nambu–Poisson manifold of order  $m$ .

The following theorem describes the local structure of the Nambu–Poisson brackets of order  $n$ , with  $n \geq 3$ .

**Theorem 2.2.** [1, 11, 14, 22, 23] Let  $M$  be a differentiable manifold of dimension  $m$ . The  $n$ -vector  $\Lambda$ ,  $n \geq 3$ , defines a Nambu–Poisson structure on  $M$  if and only if for all  $x \in M$  where  $\Lambda(x) \neq 0$ , there exist local coordinates  $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$  around  $x$  such that

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}.$$

**Remark 2.3.** A point  $x$  of a Nambu–Poisson manifold  $(M, \Lambda)$  of order  $n \geq 3$  is said to be regular if  $\Lambda(x) \neq 0$ .

**Remark 2.4.** Let  $(M, \Lambda)$  be a Nambu–Poisson manifold of order  $n$ , with  $n \geq 3$ , and consider the characteristic distribution  $\mathcal{D}$  on  $M$  given by

$$x \in M \longrightarrow \mathcal{D}(x) = \#(\Lambda^{n-1} T_x^* M) = \langle \{X_{f_1 \dots f_{n-1}}(x)/f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})\} \rangle \subseteq T_x M.$$

Then,  $\mathcal{D}$  defines a generalized foliation on  $M$  whose leaves are either points or  $n$ -dimensional manifolds endowed with a Nambu–Poisson structure coming from a volume form (see [14]).

### 2.2. Leibniz algebras and cohomology

First, we recall the definition of real Leibniz algebra (see [17–19]).

A Leibniz algebra structure on a real vector space  $\mathfrak{g}$  is a  $\mathbb{R}$ -bilinear map  $\{ , \} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying the Leibniz identity, that is,

$$\{a_1, \{a_2, a_3\}\} - \{\{a_1, a_2\}, a_3\} - \{a_2, \{a_1, a_3\}\} = 0 \tag{2.9}$$

for  $a_1, a_2, a_3 \in \mathfrak{g}$ . In such a case, one says that  $(\mathfrak{g}, \{ , \})$  is a Leibniz algebra.

Moreover, if the skew-symmetric condition is required then  $(\mathfrak{g}, \{ , \})$  is a Lie algebra. In this sense, a Leibniz algebra is a non-commutative version of a Lie algebra.

Let  $(\mathfrak{g}, \{ , \})$  be a Leibniz algebra and  $\mathcal{M}$  be a real vector space endowed with a  $\mathbb{R}$ -bilinear map

$$\mathfrak{g} \times \mathcal{M} \longrightarrow \mathcal{M}$$

such that  $\{a_1, a_2\}m = a_1(a_2m) - a_2(a_1m)$ , for all  $a_1, a_2 \in \mathfrak{g}$  and  $m \in \mathcal{M}$ . Then  $\mathcal{M}$  is a  $\mathfrak{g}$ -module relative to the representation of  $\mathfrak{g}$  on  $\mathcal{M}$ .

If  $\mathcal{M}$  is a  $\mathfrak{g}$ -module we can introduce a cohomology complex as follows.

A  $k$ -linear mapping  $c^k : \mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow \mathcal{M}$  is called a  $\mathcal{M}$ -valued  $k$ -cochain. We denote by  $C^k(\mathfrak{g}; \mathcal{M})$  the real vector space of these cochains.

The operator  $\partial^k : C^k(\mathfrak{g}; \mathcal{M}) \longrightarrow C^{k+1}(\mathfrak{g}; \mathcal{M})$  given by

$$\begin{aligned} \partial^k c^k(a_0, \dots, a_k) &= \sum_{i=0}^k (-1)^i a_i c^k(a_0, \dots, \widehat{a}_i, \dots, a_k) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i-1} c^k(a_0, \dots, \widehat{a}_i, \dots, a_{j-1}, \{a_i, a_j\}, a_{j+1}, \dots, a_k) \end{aligned} \tag{2.10}$$

defines a coboundary since  $\partial^{k+1} \circ \partial^k = 0$ . Hence,  $(C^*(\mathfrak{g}; \mathcal{M}), \partial)$  is a cohomology complex and the corresponding cohomology spaces

$$H^k(\mathfrak{g}; \mathcal{M}) = \frac{\ker\{\partial^k : C^k(\mathfrak{g}; \mathcal{M}) \rightarrow C^{k+1}(\mathfrak{g}; \mathcal{M})\}}{\text{Im}\{\partial^{k-1} : C^{k-1}(\mathfrak{g}; \mathcal{M}) \rightarrow C^k(\mathfrak{g}; \mathcal{M})\}}$$

are called the Leibniz cohomology groups of  $\mathfrak{g}$  with coefficients in  $\mathcal{M}$  (see [17–19]).

Note that if  $(\mathfrak{g}, \{ , \})$  is a Lie algebra and  $c^k$  is a skew-symmetric  $\mathcal{M}$ -valued  $k$ -cochain, then  $\partial^k c^k$  is a skew-symmetric  $\mathcal{M}$ -valued  $(k + 1)$ -cochain. Thus, we can consider the subcomplex

$(C_{Lie}^*(\mathfrak{g}; \mathcal{M}), \partial)$  of  $(C^*(\mathfrak{g}; \mathcal{M}), \partial)$  that consists of the skew-symmetric  $\mathcal{M}$ -valued cochains. In fact, the cohomology of this subcomplex is just the cohomology  $H_{Lie}^*(\mathfrak{g}; \mathcal{M})$  of the Lie algebra  $\mathfrak{g}$  with coefficients in  $\mathcal{M}$ . Therefore, we have defined a natural homomorphism

$$i^k : H_{Lie}^k(\mathfrak{g}; \mathcal{M}) \rightarrow H^k(\mathfrak{g}; \mathcal{M})$$

between the cohomology groups  $H_{Lie}^k(\mathfrak{g}; \mathcal{M})$  and  $H^k(\mathfrak{g}; \mathcal{M})$ .

**Examples 2.5.** (a) Let  $M$  be a differentiable manifold and  $(\mathfrak{X}(M), [\cdot, \cdot])$  be the Lie algebra of the vector fields on  $M$ . Then, the real vector space  $C^\infty(M, \mathbb{R})$  is a  $\mathfrak{X}(M)$ -module with the usual multiplication

$$\mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}) \quad (X, f) \mapsto X(f).$$

The  $k$ -cochains in the Leibniz cohomology complex are the  $k$ -linear mappings  $c^k : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow C^\infty(M, \mathbb{R})$  and the Leibniz cohomology operator  $d : C_{Leib}^k(M) = C^k(\mathfrak{X}(M); C^\infty(M, \mathbb{R})) \rightarrow C_{Leib}^{k+1}(M) = C^{k+1}(\mathfrak{X}(M); C^\infty(M, \mathbb{R}))$  is defined as the exterior differential operator, that is,

$$\begin{aligned} dc^k(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i X_i(c^k(X_0, \dots, \widehat{X}_i, \dots, X_k)) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i-1} c^k(X_0, \dots, \widehat{X}_i, \dots, X_{j-1}, [X_i, X_j], X_{j+1}, \dots, X_k) \end{aligned} \quad (2.11)$$

for all  $X_0, \dots, X_k \in \mathfrak{X}(M)$ .

The resultant cohomology  $H^*(\mathfrak{X}(M); C^\infty(M, \mathbb{R}))$  is the *Leibniz cohomology* of  $M$  and it is denoted by  $H_{Leib}^*(M)$  (for a detailed study of this cohomology, we refer to [20]). Note that the de Rham cohomology of  $M$ ,  $H_{dR}^*(M)$ , is just the cohomology of the subcomplex of the skew-symmetric  $C^\infty(M, \mathbb{R})$ -valued cochains that are  $C^\infty(M, \mathbb{R})$ -linear.

(b) Let  $(M, \Lambda)$  be a Nambu–Poisson manifold of order  $n$  with Nambu–Poisson bracket  $\{\cdot, \dots, \cdot\}$ . Consider on  $\bigwedge^{n-1}(C^\infty(M, \mathbb{R}))$  the bracket  $\{\cdot, \cdot\}'$  characterized by the formula

$$\{f_1 \wedge \dots \wedge f_{n-1}, g_1 \wedge \dots \wedge g_{n-1}\}' = \sum_{i=1}^{n-1} g_1 \wedge \dots \wedge \{f_1, \dots, f_{n-1}, g_i\} \wedge \dots \wedge g_{n-1} \quad (2.12)$$

for  $f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1} \in C^\infty(M, \mathbb{R})$ . Using equation (2.7), we deduce that  $(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})), \{\cdot, \cdot\}')$  is a Leibniz algebra (see [7, 11, 23]). Moreover, the real vector space  $C^\infty(M, \mathbb{R})$  is a  $\bigwedge^{n-1}(C^\infty(M, \mathbb{R}))$ -module with the multiplication

$$\bigwedge^{n-1}(C^\infty(M, \mathbb{R})) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$$

characterized by

$$(f_1 \wedge \dots \wedge f_{n-1}, f) \mapsto X_{f_1 \dots f_{n-1}}(f). \quad (2.13)$$

The resultant cohomology

$$H^*(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$$

was studied in [7, 11].

### 3. Leibniz algebroids and Nambu–Poisson manifolds

In this section we will define a generalization of the notion of the Lie algebroid (the Leibniz algebroid) and we will prove that a Nambu–Poisson manifold has an associated structure of this type.

**Definition 3.1.** A Leibniz algebroid structure on a differentiable vector bundle  $\pi : E \rightarrow M$  is a pair that consists of a Leibniz algebra structure  $[\![ \ , \ ]\!]$  on the space  $\Gamma(E)$  of the global cross sections of  $\pi : E \rightarrow M$  and a vector bundle morphism  $\varrho : E \rightarrow TM$ , called the anchor map, such that the induced map  $\varrho : \Gamma(E) \rightarrow \Gamma(TM) = \mathfrak{X}(M)$  satisfies the following relations:

- (a)  $\varrho[\![s_1, s_2]\!] = [\varrho(s_1), \varrho(s_2)]$ ,
- (b)  $[\![s_1, fs_2]\!] = f[\![s_1, s_2]\!] + \varrho(s_1)(f)s_2$ ,

for all  $s_1, s_2 \in \Gamma(E)$  and  $f \in C^\infty(M, \mathbb{R})$ .

A triple  $(E, [\![ \ , \ ]\!], \varrho)$  is called a Leibniz algebroid over  $M$ .

**Remark 3.2.** (a) If  $(\mathfrak{g}, \{ \ , \ }, \varrho \equiv 0)$  is a Leibniz algebra then  $(\mathfrak{g}, \{ \ , \ }, \varrho \equiv 0)$  is a Leibniz algebroid over a point.

(b) Every Lie algebroid over a manifold  $M$  is trivially a Leibniz algebroid. In fact, a Leibniz algebroid  $(E, [\![ \ , \ ]\!], \varrho)$  over  $M$  is a Lie algebroid if and only if the Leibniz bracket  $[\![ \ , \ ]\!]$  on  $\Gamma(E)$  is skew-symmetric.

If  $(M, \Lambda)$  is a Poisson manifold then it is possible to define a bracket of 1-forms  $[\![ \ , \ ]\!]'$  as follows:

$$[\![\alpha, \beta]\!] = \mathcal{L}_{\#(\alpha)}\beta - \mathcal{L}_{\#(\beta)}\alpha - d(\Lambda(\alpha, \beta)). \tag{3.1}$$

The space of 1-forms  $\Omega^1(M)$  endowed with this bracket is a Lie algebra in such a sense that the triple  $(T^*M, [\![ \ , \ ]\!]', \#)$  is a Lie algebroid over  $M$ , where  $T^*M$  is the cotangent bundle of  $M$  and  $\# : T^*M \rightarrow TM$  is the homomorphism of vector bundles given by (2.4) (see [3, 26]).

Next, we will prove that associated to a Nambu–Poisson manifold of order  $n$ , with  $n \geq 3$ , there is a canonical Leibniz algebroid.

Let  $(M, \Lambda)$  be an  $m$ -dimensional Nambu–Poisson manifold of order  $n$ ,  $n \geq 3$ , with Nambu–Poisson bracket  $\{ \ , \dots, \}$ .

**Proposition 3.3.** For all  $\alpha, \beta \in \Omega^{n-1}(M)$  we have

$$[\#\alpha, \#\beta] = \#(\mathcal{L}_{\#\alpha}\beta + (-1)^n(i(d\alpha)\Lambda)\beta)$$

where  $\# : \Omega^{n-1}(M) \rightarrow \mathfrak{X}(M)$  is the homomorphism defined in (2.3) and (2.4) and  $\mathcal{L}$  is the Lie derivative operator.

**Proof.** Using (2.3) and (2.4) we have that

$$\begin{aligned} [\#\alpha, \#\beta] &= i(\beta)(\mathcal{L}_{\#\alpha}\Lambda) + i(\mathcal{L}_{\#\alpha}\beta)\Lambda \\ &= i(\beta)(\mathcal{L}_{\#\alpha}\Lambda) + \#(\mathcal{L}_{\#\alpha}\beta). \end{aligned} \tag{3.2}$$

On the other hand,

$$\mathcal{L}_{\#\alpha}\Lambda = (-1)^n(i(d\alpha)\Lambda)\Lambda. \tag{3.3}$$

Indeed, if  $x \in M$  and  $\Lambda(x) = 0$  then  $(\mathcal{L}_{\#\alpha}\Lambda)(x) = (-1)^n(i(d\alpha)\Lambda)(x)\Lambda(x) = 0$ .



If  $x \in M$  and  $\Lambda(x) \neq 0$ , then (see theorem 2.2) there exist local coordinates  $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$  in an open subset  $U$  of  $M$ ,  $x \in U$ , such that

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}. \quad (3.4)$$

So, to prove (3.3) it suffices of course to check this formula for local  $(n - 1)$ -forms

$$\alpha = \sum_{i=1}^n \alpha_i dx_1 \wedge \dots \wedge \widehat{dx}_i \wedge \dots \wedge dx_n$$

with  $\alpha_i \in C^\infty(U, \mathbb{R})$ . Now, from (2.3), (2.4) and (3.4), one deduces that (3.3) is true for this type of  $(n - 1)$ -forms.

Finally, using (2.3), (2.4), (3.2) and (3.3), the result follows.  $\square$

The above result suggests introducing the following definition.

**Definition 3.4.** Let  $(M, \Lambda)$  be an  $m$ -dimensional Nambu–Poisson manifold of order  $n$ , with  $3 \leq n \leq m$ . The bracket of  $(n - 1)$ -forms on  $M$  is the  $\mathbb{R}$ -bilinear operation  $\llbracket \cdot, \cdot \rrbracket: \Omega^{n-1}(M) \times \Omega^{n-1}(M) \rightarrow \Omega^{n-1}(M)$  given by

$$\llbracket \alpha, \beta \rrbracket = \mathcal{L}_{\# \alpha} \beta + (-1)^n (i(d\alpha)\Lambda)\beta \quad (3.5)$$

for  $\alpha, \beta \in \Omega^{n-1}(M)$ .

**Remark 3.5.** If  $(M, \Lambda)$  is a Poisson manifold one can consider also the bracket of 1-forms  $\llbracket \cdot, \cdot \rrbracket$  given by (3.5). However, this bracket and the Lie algebroid bracket  $\llbracket \cdot, \cdot \rrbracket'$  defined by (3.1) are not directly related. The reason is that, in general, equation (3.3) does not hold for arbitrary Poisson manifolds.

The mapping  $\llbracket \cdot, \cdot \rrbracket$  is characterized as follows.

**Theorem 3.6.** Let  $(M, \Lambda)$  be an  $m$ -dimensional Nambu–Poisson manifold of order  $n$ , with  $3 \leq n \leq m$ . Then exists a unique  $\mathbb{R}$ -bilinear operation  $\llbracket \cdot, \cdot \rrbracket: \Omega^{n-1}(M) \times \Omega^{n-1}(M) \rightarrow \Omega^{n-1}(M)$  such that:

(a) For all  $f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1} \in C^\infty(M, \mathbb{R})$ , we have

$$\begin{aligned} \llbracket df_1 \wedge \dots \wedge df_{n-1}, dg_1 \wedge \dots \wedge dg_{n-1} \rrbracket \\ = \sum_{i=1}^{n-1} dg_1 \wedge \dots \wedge d\{f_1, \dots, f_{n-1}, g_i\} \wedge \dots \wedge dg_{n-1}. \end{aligned} \quad (3.6)$$

(b) For all  $f \in C^\infty(M, \mathbb{R})$  and  $\alpha, \beta \in \Omega^{n-1}(M)$ , we have

$$\llbracket \alpha, f\beta \rrbracket = f\llbracket \alpha, \beta \rrbracket + \# \alpha(f)\beta \quad (3.7)$$

$$\llbracket f\alpha, \beta \rrbracket = f\llbracket \alpha, \beta \rrbracket - i(\# \alpha)(df \wedge \beta). \quad (3.8)$$

This operation is given by (3.5).

**Proof.** It is easy to prove that the bracket defined in (3.5) satisfies (3.6)–(3.8).

Now, suppose that  $\llbracket \cdot, \cdot \rrbracket_1: \Omega^{n-1}(M) \times \Omega^{n-1}(M) \rightarrow \Omega^{n-1}(M)$  is a  $\mathbb{R}$ -bilinear operation which satisfies (3.6)–(3.8). Then  $\llbracket \cdot, \cdot \rrbracket_1$  must be of local type, i.e.  $\llbracket \alpha, \beta \rrbracket_1(x_0)$  will depend on  $\alpha$  and  $\beta$  around  $x_0$  only, for all  $x_0 \in M$ . Indeed, if  $\beta_{1|U} = \beta_{2|U}$  for an open neighbourhood  $U$

of  $x_0$ , and if  $f$  is a  $C^\infty$  real-valued function that vanishes outside  $U$ , and equals 1 on a compact neighbourhood  $V_{x_0} \subseteq U$ , then  $\sigma = f\beta_1 = f\beta_2$  is well defined on  $M$  and, by (3.7) we have

$$\llbracket \alpha, \sigma \rrbracket_1(x_0) = \llbracket \alpha, \beta_1 \rrbracket_1(x_0) \quad \llbracket \alpha, \sigma \rrbracket_1(x_0) = \llbracket \alpha, \beta_2 \rrbracket_1(x_0)$$

i.e.  $\llbracket \alpha, \beta_1 \rrbracket_1(x_0) = \llbracket \alpha, \beta_2 \rrbracket_1(x_0)$ .

Similarly, if  $\alpha_{1|U} = \alpha_{2|U}$  and  $\nu = f\alpha_1 = f\alpha_2$  then, from (3.8), we deduce that

$$\llbracket \nu, \beta \rrbracket_1(x_0) = \llbracket \alpha_1, \beta \rrbracket_1(x_0) \quad \llbracket \nu, \beta \rrbracket_1(x_0) = \llbracket \alpha_2, \beta \rrbracket_1(x_0)$$

that is,  $\llbracket \alpha_1, \beta \rrbracket_1(x_0) = \llbracket \alpha_2, \beta \rrbracket_1(x_0)$ .

Next, we will show that  $\llbracket \cdot, \cdot \rrbracket_1 = \llbracket \cdot, \cdot \rrbracket$ .

Let  $x$  be a point of  $M$  and  $\alpha$  and  $\beta$  ( $n - 1$ )-forms on  $M$ .

Assume that  $(x^1, \dots, x^m)$  are local coordinates in an open neighbourhood  $U$  of  $x$  and that in  $U$  we have

$$\begin{aligned} \alpha &= \sum_{1 \leq i_1 < \dots < i_{n-1} \leq m} \alpha_{i_1 \dots i_{n-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}} \\ \beta &= \sum_{1 \leq j_1 < \dots < j_{n-1} \leq m} \beta_{j_1 \dots j_{n-1}} dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}}. \end{aligned}$$

Using (2.5), (3.5)–(3.8) and the local character of the bracket  $\llbracket \cdot, \cdot \rrbracket_1$ , we obtain that

$$\begin{aligned} \llbracket \alpha, \beta \rrbracket_1(x) &= \sum_{\substack{1 \leq i_1 < \dots < i_{n-1} \leq m \\ 1 \leq j_1 < \dots < j_{n-1} \leq m}} \left[ \sum_{k=1, \dots, n-1} (\alpha_{i_1 \dots i_{n-1}} \beta_{j_1 \dots j_{n-1}} dx^{j_1} \wedge \dots \wedge dx^{j_{k-1}} \right. \\ &\quad \wedge d\{x^{i_1}, \dots, x^{i_{n-1}}, x^{j_k}\} \wedge dx^{j_{k+1}} \wedge \dots \wedge dx^{j_{n-1}}) \\ &\quad - \beta_{j_1 \dots j_{n-1}} X_{x^{i_1} \dots x^{i_{n-1}}}(\alpha_{i_1 \dots i_{n-1}}) dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}} \\ &\quad + \beta_{j_1 \dots j_{n-1}} d\alpha_{i_1 \dots i_{n-1}} \wedge i_{X_{x^{i_1} \dots x^{i_{n-1}}}}(dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}}) \\ &\quad \left. + \alpha_{i_1 \dots i_{n-1}} X_{x^{j_1} \dots x^{j_{n-1}}}(\beta_{j_1 \dots j_{n-1}}) dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}} \right] (x) \\ &= \llbracket \alpha, \beta \rrbracket(x). \end{aligned}$$

From the arbitrariness of the point  $x$ , it follows that  $\llbracket \alpha, \beta \rrbracket_1 = \llbracket \alpha, \beta \rrbracket$ . □

Now, we will prove that a Nambu–Poisson manifold of order  $n$ , with  $n \geq 3$ , has an associated Leibniz algebroid.

**Theorem 3.7.** *Let  $(M, \Lambda)$  be an  $m$ -dimensional Nambu–Poisson manifold of order  $n$ , with  $3 \leq n \leq m$ . Then, the triple  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$  is a Leibniz algebroid over  $M$ , where  $\llbracket \cdot, \cdot \rrbracket : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \rightarrow \Omega^{n-1}(M)$  is the bracket of  $(n - 1)$ -forms defined by (3.5) and  $\# : \wedge^{n-1}(T^*M) \rightarrow TM$  is the homomorphism of vector bundles given by (2.3).*

**Proof.** We must prove that

$$\llbracket \alpha, \llbracket \beta, \gamma \rrbracket \rrbracket - \llbracket \llbracket \alpha, \beta \rrbracket, \gamma \rrbracket - \llbracket \beta, \llbracket \alpha, \gamma \rrbracket \rrbracket = 0 \tag{3.9}$$

for  $\alpha, \beta, \gamma \in \Omega^{n-1}(M)$ .

From (3.3), (3.5) and proposition 3.3, we obtain that

$$(i(d\llbracket \alpha, \beta \rrbracket)\Lambda - \#\alpha(i(d\beta)\Lambda) + \#\beta(i(d\alpha)\Lambda))\Lambda = 0.$$

Thus,

$$i(d\llbracket\alpha, \beta\rrbracket)\Lambda = \#\alpha(i(d\beta)\Lambda) - \#\beta(i(d\alpha)\Lambda). \quad (3.10)$$

On the other hand, using (3.5), we have that

$$\llbracket\alpha, \llbracket\beta, \gamma\rrbracket\rrbracket - \llbracket\beta, \llbracket\alpha, \gamma\rrbracket\rrbracket = \mathcal{L}_{\llbracket\#\alpha, \#\beta\rrbracket}\gamma + (-1)^n(\#\alpha(i(d\beta)\Lambda) - \#\beta(i(d\alpha)\Lambda))\gamma$$

which implies that (see (3.10) and proposition 3.3)

$$\llbracket\alpha, \llbracket\beta, \gamma\rrbracket\rrbracket - \llbracket\beta, \llbracket\alpha, \gamma\rrbracket\rrbracket = \mathcal{L}_{\llbracket\#\alpha, \#\beta\rrbracket}\gamma + (-1)^n(i(d\llbracket\alpha, \beta\rrbracket)\Lambda)\gamma. \quad (3.11)$$

Therefore, from (3.5) and (3.11), it follows that (3.9) holds. Hence, we deduce that the bracket  $\llbracket \cdot, \cdot \rrbracket$  induces a Leibniz algebra structure on  $\Omega^{n-1}(M)$ .

Using this fact, equation (3.7) and proposition 3.3, we conclude that the triple  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$  is a Leibniz algebroid over  $M$ .  $\square$

**Remark 3.8.** In [13] the authors have introduced the notion of a Filippov algebroid, as a  $n$ -ary generalization of Lie algebroids. Indeed, the binary bracket of sections in a Lie algebroid is replaced in a Filippov algebroid  $E \rightarrow M$  by an  $n$ -bracket  $\llbracket \cdot, \dots, \cdot \rrbracket$  on  $\Gamma(E)$  satisfying the fundamental identity and, the anchor map is a vector bundle morphism  $\Lambda^{n-1}(E) \rightarrow TM$  compatible with the  $n$ -bracket. In [13, 27] an  $n$ -bracket of 1-forms on a Nambu–Poisson manifold is defined. However, this bracket does not satisfy the fundamental identity.

In general, the bracket defined in (3.5) is not skew-symmetric and consequently the Leibniz algebroid  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$  is not a Lie algebroid.

In the following result we characterize when this Leibniz algebroid is a Lie algebroid on an oriented manifold.

**Theorem 3.9.** *Let  $M$  be an oriented manifold of dimension  $m$ ,  $m \geq 3$ . The unique non-null Nambu–Poisson structures of order greater than two on  $M$  such that the Leibniz algebroid is a Lie algebroid are those defined by non-null  $m$ -vectors.*

**Proof.** Suppose that  $\Lambda$  is a non-null  $m$ -vector. Then  $(M, \Lambda)$  is a Nambu–Poisson manifold of order  $m$  (see examples 2.1).

Now, if  $\alpha$  and  $\beta$  are  $(m-1)$ -forms on  $M$ , we consider the  $(m-1)$ -form  $\sigma$  on  $M$  defined by

$$\sigma = \llbracket\alpha, \beta\rrbracket + \llbracket\beta, \alpha\rrbracket.$$

We must prove that  $\sigma = 0$ .

Since the set

$$A = \{x \in M / \Lambda(x) \neq 0\}$$

is an open subset of  $M$ ,  $\Lambda$  induces a Nambu–Poisson structure  $\Lambda_A$  on  $A$  of order  $m$  which is non-null at every point. Then, as  $M$  is oriented, we deduce that  $\Lambda_A$  is defined by a volume form on  $A$  and the corresponding homomorphism

$$\#_A : \Omega^{n-1}(A) \rightarrow \mathfrak{X}(A)$$

given by (2.3) and (2.4), is an isomorphism. Using this last fact, proposition 3.3 and the skew-symmetry of the Lie bracket  $[\cdot, \cdot]$  of vector fields, we obtain that  $\sigma = 0$  on  $A$ .

On the other hand, it is obvious that  $\sigma$  is null on the exterior of  $A$  (note that the exterior of  $A$  is an open subset of  $M$  and that  $\Lambda = 0$  on such a set). Finally, by continuity we conclude that  $\sigma = 0$  on the boundary of  $A$ . Thus,  $\sigma = 0$  on  $M$ .

Conversely, suppose that  $(M, \Lambda)$  is an oriented  $m$ -dimensional Nambu–Poisson manifold of order  $n$ , with  $3 \leq n \leq m$  and that  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$  is a Lie algebroid.

Since  $\Lambda$  is a non-null  $n$ -vector, there exists a point of  $M$  such that  $\Lambda(x) \neq 0$  and there exist local coordinates  $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$  on an open neighbourhood  $U$  of  $x$  such that the  $n$ -vector  $\Lambda_U$  induced by  $\Lambda$  on  $U$  is given by (see theorem 2.2)

$$\Lambda_U = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}.$$

Using the fact that  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$  is a Lie algebroid, we deduce that the bracket  $\llbracket \cdot, \cdot \rrbracket_U : \Omega^{n-1}(U) \times \Omega^{n-1}(U) \rightarrow \Omega^{n-1}(U)$  defined by  $\Lambda_U$  (see equation (3.5)) is skew-symmetric.

Now, if  $n < m$  we can consider the  $(n - 1)$ -forms on  $U$  given by

$$\begin{aligned} \alpha &= dx^1 \wedge \dots \wedge dx^{n-3} \wedge dx^n \wedge dx^{n+1} \\ \beta &= x^{n-2} dx^1 \wedge \dots \wedge dx^{n-1} \end{aligned}$$

and a direct computation proves that  $0 = \llbracket \alpha, \beta \rrbracket_U \neq -\llbracket \beta, \alpha \rrbracket_U$ . This is a contradiction. Hence,  $n = m$ . □

From theorem 3.9, we obtain:

**Corollary 3.10.** *Let  $M$  be an oriented manifold of dimension  $m$ ,  $m \geq 3$ , and  $v$  be a volume form on  $M$ . Then the Leibniz algebroid associated with the Nambu–Poisson manifold  $(M, \Lambda_v)$  is a Lie algebroid.*

#### 4. Cohomology of a Leibniz algebroid and the modular class of a Nambu–Poisson manifold

Let  $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$  be a Leibniz algebroid over a manifold  $M$ . From definition 3.1, we deduce that  $C^\infty(M, \mathbb{R})$  is a  $\Gamma(E)$ -module with the multiplication

$$\Gamma(E) \times C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R}) \quad (s, f) \mapsto \varrho(s)(f). \tag{4.1}$$

Thus, we can consider the differential complex  $(C^*(\Gamma(E); C^\infty(M, \mathbb{R})), \partial)$  and its cohomology  $H^*(\Gamma(E); C^\infty(M, \mathbb{R}))$ , that is, the cohomology of  $\Gamma(E)$  with coefficients in  $C^\infty(M, \mathbb{R})$  (see section 2.2).  $H^*(\Gamma(E); C^\infty(M, \mathbb{R}))$  is called *the Leibniz algebroid cohomology of  $E$* . Using (2.10), we have that

$$\begin{aligned} \partial^k c^k(s_0, \dots, s_k) &= \sum_{i=0}^k (-1)^i \varrho(s_i)(c^k(s_0, \dots, \widehat{s}_i, \dots, s_k)) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i-1} c^k(s_0, \dots, \widehat{s}_i, \dots, s_{j-1}, \llbracket s_i, s_j \rrbracket, s_{j+1}, \dots, s_k) \end{aligned} \tag{4.2}$$

for  $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$  and  $s_0, \dots, s_k \in \Gamma(E)$ .

**Remark 4.1.** (a) Let  $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$  be a Lie algebroid over  $M$  and  $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$ . If  $c^k$  is skew-symmetric and  $C^\infty(M, \mathbb{R})$ -linear then  $\partial^k c^k$  is also skew-symmetric and  $C^\infty(M, \mathbb{R})$ -linear.

The *Lie algebroid cohomology* of  $E$  is the cohomology of the subcomplex of the cochains which are skew-symmetric and  $C^\infty(M, \mathbb{R})$ -linear (see [21]).

- (b) If  $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$  is a Leibniz algebroid over  $M$  and  $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$  is skew-symmetric (respectively,  $C^\infty(M, \mathbb{R})$ -linear) then, in general,  $\partial^k c^k$  is not skew-symmetric (respectively,  $C^\infty(M, \mathbb{R})$ -linear).

The following result relates the Leibniz algebroid cohomology of  $E$  with the Leibniz cohomology of the base manifold  $M$ .

**Proposition 4.2.** *Let  $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$  be a Leibniz algebroid over a manifold  $M$ . Suppose that  $(C_{Leib}^*(M), d)$  is the Leibniz cohomology complex of the manifold  $M$  and denote by*

$$\tilde{\varrho}^k : C_{Leib}^k(M) \longrightarrow C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$$

the homomorphism defined by

$$\tilde{\varrho}^k(c^k)(s_1, \dots, s_k) = c^k(\varrho(s_1), \dots, \varrho(s_k))$$

for  $c^k \in C_{Leib}^k(M)$  and  $s_1, \dots, s_k \in \Gamma(E)$ . Then, the mappings  $\tilde{\varrho}^k$  induce a homomorphism of complexes

$$\tilde{\varrho} : (C_{Leib}^*(M), d) \longrightarrow (C^*(\Gamma(E); C^\infty(M, \mathbb{R})), \partial).$$

Therefore, we have the corresponding homomorphism in cohomology

$$\tilde{\varrho} : H_{Leib}^*(M) \longrightarrow H^*(\Gamma(E); C^\infty(M, \mathbb{R})).$$

**Proof.** It follows using (2.11), (4.2) and definition 3.1. □

**Remark 4.3.** In fact, if  $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$  is a Lie algebroid over  $M$ , we can define a homomorphism  $\bar{\varrho}$  between the de Rham cohomology of  $M$ ,  $H_{dR}^*(M)$ , and the Leibniz algebroid cohomology of  $E$  given by

$$\bar{\varrho} = \tilde{\varrho} \circ i : H_{dR}^*(M) \longrightarrow H_{Leib}^*(M) \longrightarrow H^*(\Gamma(E); C^\infty(M, \mathbb{R}))$$

where  $i : H_{dR}^*(M) \longrightarrow H_{Leib}^*(M)$  is the homomorphism induced by the natural inclusion.

Using proposition 4.2, we have

**Corollary 4.4.** *Let  $(M, \Lambda)$  be a Nambu–Poisson manifold of order  $n$ , with  $n \geq 3$ , and let  $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$  be the Leibniz algebroid associated with  $M$ . Suppose that*

$$\tilde{\#}^k : C_{Leib}^k(M) \longrightarrow C^k(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$$

is the homomorphism defined by

$$\tilde{\#}^k(c^k)(\alpha_1, \dots, \alpha_k) = c^k(\#\alpha_1, \dots, \#\alpha_k)$$

for  $c^k \in C_{Leib}^k(M)$  and  $\alpha_1, \dots, \alpha_k \in \Omega^{n-1}(M)$ . Then, the mappings  $\tilde{\#}^k$  induce a homomorphism of complexes

$$\tilde{\#} : (C_{Leib}^*(M), d) \longrightarrow (C^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})), \partial).$$

Therefore, we have the corresponding homomorphism in cohomology

$$\tilde{\#} : H_{Leib}^*(M) \longrightarrow H^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})).$$

For the particular case of a Nambu–Poisson structure coming from a volume form, we deduce the following result:

**Proposition 4.5.** *Let  $M$  be an oriented manifold of dimension  $m$ , with  $m \geq 3$ , and let  $\nu$  be a volume form on  $M$ . Then the Leibniz cohomology of the algebroid associated with  $(M, \Lambda_\nu)$  is isomorphic to the Leibniz cohomology of  $M$ .*

**Proof.** Since  $\nu$  is a volume form, the homomorphism

$$\# : \Omega^{n-1}(M) \longrightarrow \mathfrak{X}(M)$$

defined by (2.3) and (2.4), is an isomorphism.

Using this fact and corollary 4.4 the result follows.  $\square$

**Remark 4.6.** Note that the Leibniz algebroid  $(\bigwedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$  associated with  $(M, \Lambda_\nu)$  is also a Lie algebroid (see corollary 3.10) and that the Lie algebroid cohomology of  $\bigwedge^{n-1}(T^*M)$  is isomorphic to the de Rham cohomology of  $M$ .

For a Nambu–Poisson manifold  $(M, \Lambda)$  of order  $n$ , we denote by  $\{ \cdot, \cdot \}'$  the Leibniz bracket on  $\bigwedge^{n-1}(C^\infty(M, \mathbb{R}))$  characterized by (2.12). Then, the real vector space  $C^\infty(M, \mathbb{R})$  is a  $\bigwedge^{n-1}(C^\infty(M, \mathbb{R}))$ -module with the multiplication given by (2.13). Thus, we can consider the corresponding differential complex  $(C^*(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R})), \partial')$  and its cohomology  $H^*(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$ .

In the next result, we obtain a relation between the Leibniz algebroid cohomology of  $\bigwedge^{n-1}(T^*M)$  and the cohomology  $H^*(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$ .

**Proposition 4.7.** *Let  $(M, \Lambda)$  be a Nambu–Poisson manifold of order  $n$ , with  $n \geq 3$ . Then the mapping*

$$\Phi : \bigwedge^{n-1}(C^\infty(M, \mathbb{R})) \longrightarrow \Omega^{n-1}(M) \quad f_1 \wedge \dots \wedge f_{n-1} \mapsto df_1 \wedge \dots \wedge df_{n-1} \quad (4.3)$$

induces a natural homomorphism of complexes

$$\tilde{\Phi} : (C^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})), \partial) \longrightarrow (C^*(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R})), \partial')$$

and therefore we have the corresponding homomorphism in cohomology

$$\tilde{\Phi} : H^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})) \longrightarrow H^*(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R})).$$

**Proof.** Consider the mappings

$$\tilde{\Phi}^k : C^k(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})) \longrightarrow C^k(\bigwedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$$

defined by

$$\tilde{\Phi}^k(c^k)(F_1, \dots, F_k) = c^k(\Phi(F_1), \dots, \Phi(F_k)) \quad (4.4)$$

for  $F_1, \dots, F_k \in \bigwedge^{n-1}(C^\infty(M, \mathbb{R}))$ .

From (2.12), (3.6) and (4.3), we obtain

$$\Phi(\{F_i, F_j\}') = \llbracket \Phi(F_i), \Phi(F_j) \rrbracket.$$

Using this fact, equations (2.5), (2.10), (2.13), (4.2) and (4.4), we obtain that the mappings  $\tilde{\Phi}^k$  induce a homomorphism of complexes.  $\square$

Next, we will introduce the modular class of an oriented Nambu–Poisson manifold. For this purpose, we prove the following result.

**Theorem 4.8.** Let  $(M, \Lambda)$  be an oriented  $m$ -dimensional Nambu–Poisson manifold of order  $n$ , with  $n \geq 3$ , and  $\nu$  be a volume form. Consider the mapping

$$\mathcal{M}_\nu : C^\infty(M, \mathbb{R}) \times \dots^{(n-1)} \dots \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$$

defined by

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \nu = \mathcal{M}_\nu(f_1, \dots, f_{n-1})\nu \quad (4.5)$$

for  $f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})$ . Then:

- (a)  $\mathcal{M}_\nu$  is a skew-symmetric  $(n-1)$ -linear mapping and a derivation in each argument with respect to the usual product of functions. Thus,  $\mathcal{M}_\nu$  defines an  $(n-1)$ -vector on  $M$ .  
 (b) The mapping

$$\mathcal{M}_\Lambda : \Omega^{n-1}(M) \longrightarrow C^\infty(M, \mathbb{R}) \quad \alpha \mapsto i(\alpha)\mathcal{M}_\nu \quad (4.6)$$

defines a 1-cocycle in the Leibniz cohomology complex associated with the Leibniz algebroid  $(\bigwedge^{n-1}(T^*M), \llbracket, \rrbracket, \#)$ .

- (c) The cohomology class  $[\mathcal{M}_\Lambda] \in H^1(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$  does not depend on the chosen volume form.

**Proof.**

- (a) It follows using (2.1), (2.2), (2.5), (2.6) and (4.5).  
 (b) We will show that for all  $\alpha \in \Omega^{n-1}(M)$  we have

$$\mathcal{L}_{\#\alpha} \nu = [i(\alpha)\mathcal{M}_\nu + (-1)^{n-1}(i(d\alpha)\Lambda)]\nu. \quad (4.7)$$

Indeed, suppose that  $\alpha = f df_1 \wedge \dots \wedge df_{n-1}$ , with  $f, f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})$ .

A direct computation proves that

$$\begin{aligned} \mathcal{L}_{\#\alpha} \nu &= df \wedge i_{X_{f_1 \dots f_{n-1}}} \nu + f \mathcal{M}_\nu(f_1, \dots, f_{n-1})\nu \\ &= df \wedge i_{X_{f_1 \dots f_{n-1}}} \nu + (i(\alpha)\mathcal{M}_\nu)\nu. \end{aligned} \quad (4.8)$$

Now, since  $i_{X_{f_1 \dots f_{n-1}}}(df \wedge \nu) = 0$ , we deduce that

$$df \wedge i_{X_{f_1 \dots f_{n-1}}} \nu = X_{f_1 \dots f_{n-1}}(f)\nu.$$

Adding this formula to (4.8) we obtain that (4.7) holds for  $\alpha = f df_1 \wedge \dots \wedge df_{n-1}$ . However, this implies that (4.7) holds for all  $\alpha \in \Omega^{n-1}(M)$ .

Using (3.10), (4.7) and theorem 3.7, we have that

$$i(\llbracket \alpha, \beta \rrbracket)\mathcal{M}_\nu = \mathcal{L}_{\#\llbracket \alpha, \beta \rrbracket} \nu + (-1)^n(i(d\llbracket \alpha, \beta \rrbracket)\Lambda)\nu = \#\alpha(i(\beta)\mathcal{M}_\nu) - \#\beta(i(\alpha)\mathcal{M}_\nu).$$

This proves (b) (see (4.2)).

- (c) Let  $\nu'$  be another volume form on  $M$ . Then there exists  $f \in C^\infty(M, \mathbb{R})$ ,  $f \neq 0$  at every point, such that  $\nu' = f\nu$ . We can suppose, without the loss of generality, that  $f > 0$ . A direct computation, using (4.7), shows that for all  $\alpha \in \Omega^{n-1}(M)$

$$i(\alpha)\mathcal{M}_{\nu'} = i(\alpha)\mathcal{M}_\nu + \#\alpha(\ln f)$$

which implies that (see (4.2))

$$\mathcal{M}_{\nu'} = \mathcal{M}_\nu + \partial(\ln f). \quad \square$$

Theorem 4.8 allows us to introduce the following definition.

**Definition 4.9.** Let  $(M, \Lambda)$  be an oriented Nambu–Poisson manifold of order  $n$ , with  $n \geq 3$ , and  $\mathcal{M}_\Lambda$  be the cocycle defined by (4.6). The cohomology class

$$[\mathcal{M}_\Lambda] \in H^1(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$$

is called the modular class of  $(M, \Lambda)$ .

**Remark 4.10.** Definition 4.9 extends for Nambu–Poisson manifolds of order greater than two the notion of modular class of a Poisson manifold introduced by Weinstein [29] (see also [4]).

For a Nambu–Poisson structure induced by a volume form, we deduce:

**Proposition 4.11.** Let  $M$  be an oriented  $m$ -dimensional manifold and  $\nu$  a volume form on  $M$ . Then the modular class of  $(M, \Lambda_\nu)$  is null.

**Proof.** Using (2.6)–(2.8), we obtain that

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \nu = 0$$

for all  $f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})$ . This implies that  $\mathcal{M}_\nu = 0$  and therefore,  $\mathcal{M}_{\Lambda_\nu} = 0$ .  $\square$

**Remark 4.12.** Suppose that  $N$  and  $L$  are oriented manifolds and let  $\nu$  be a volume form on  $N$ . The Nambu–Poisson structure  $\Lambda_\nu$  on  $N$  induces a Nambu–Poisson structure  $\Lambda$  on the product manifold  $M = N \times L$  (see examples 2.1) and from proposition 4.11, it follows that the modular class of  $(M, \Lambda)$  is null.

Using theorem 2.2 and remark 4.12, we have the following.

**Corollary 4.13.** Let  $M$  be an oriented  $m$ -dimensional Nambu–Poisson manifold of order  $n$ , with  $3 \leq n \leq m$ . If at a point  $x \in M$  we have  $\Lambda(x) \neq 0$ , then there exists an open neighbourhood  $U$  of  $x$  in  $M$  such that the modular class of  $(U, \Lambda_U)$  is null. Here  $\Lambda_U$  denotes the Nambu–Poisson structure induced by  $\Lambda$  on  $U$ .

The above results and the following example show that the vanishing of the modular class of a Nambu–Poisson manifold is closely related with its regularity.

**Example 4.14.** Consider on  $\mathbb{R}^3$  the 3-vector defined by

$$\Lambda = x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}$$

where  $(x^1, x^2, x^3)$  denote the usual coordinates on  $\mathbb{R}^3$ .

The 3-vector  $\Lambda$  defines a Nambu–Poisson structure of order three on  $\mathbb{R}^3$ .

Let  $\nu$  be the volume form given by

$$\nu = dx^1 \wedge dx^2 \wedge dx^3.$$

A direct computation proves that

$$X_{x^1 x^2} = x^3 \frac{\partial}{\partial x^3} \quad X_{x^1 x^3} = -x^3 \frac{\partial}{\partial x^2} \quad X_{x^2 x^3} = x^3 \frac{\partial}{\partial x^1}$$

and

$$\mathcal{L}_{X_{x^1 x^2}} \nu = \nu \quad \mathcal{L}_{X_{x^1 x^3}} \nu = \mathcal{L}_{X_{x^2 x^3}} \nu = 0.$$

Thus,  $\mathcal{M}_\nu = \partial/\partial x^1 \wedge \partial/\partial x^2$ .

Now, if the modular class of  $(\mathbb{R}^3, \Lambda)$  were null then there exists  $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$  such that

$$i(\alpha)\mathcal{M}_\nu = \partial f(\alpha)$$

for all  $\alpha \in \Omega^2(\mathbb{R}^3)$ . Taking  $\alpha = dx^1 \wedge dx^2$ , we would deduce that

$$1 = X_{x^1 x^2}(f) = x^3 \frac{\partial f}{\partial x^3}.$$

However, this is not possible. Thus  $[\mathcal{M}_\Lambda] \neq 0$ .



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